Introduction to Statistics - Homework #4

- Use Appendix C: Distribution tables (e.g. Z-table) from page 408 to 417 in the textbook if necessary.

Exercise 1

An employee at a call center takes between 0.5 and 6.5 minutes to respond to a phone call, and this response time is assumed to be uniformly distributed between 0.5 and 6.5. If the response times of 60 independent calls are observed, approximate the probability that the average response time is at least 4 minutes.

Solution: Since X is uniformly distributed between 0.5 and 6.5,

$$E(X) = \frac{a+b}{2} = 3.5, \qquad Var(X) = \frac{(b-a)^2}{12} = 3.$$

Thus,

$$E(\bar{X}) = 3.5, \qquad Var(\bar{X}) = \frac{3}{60} = 0.05.$$

With a sample size of $n = 60$, which is sufficiently large, by the Central Limit Theorem we approximate

$$\bar{X} \sim N(3.5, 0.05).$$

Hence,

$$Z = \frac{\bar{X} - 3.5}{\sqrt{0.05}} \sim N(0, 1).$$

Therefore, the desired probability is

$$P(\bar{X} \geq 4) = P\left(\frac{\bar{X} - 3.5}{\sqrt{0.05}} \geq \frac{4 - 3.5}{\sqrt{0.05}}\right) = P(Z \geq 2.2361) \approx 0.0127.$$

Exercise 2

When Team A and Team B play a game in a sport, probabilities that Team A wins or loses are 0.6 and 0.4, respectively. Each outcome gives 3 points for a win and 0 points for a loss. Let $X_1, X_2, \ldots, X_n$ denote the points earned by Team A in n independent games (assume independence across games). Answer the following questions.

(1) Find the joint probability mass function of X_1 and X_2.

Solution:

Since the games are independent, the joint probability distribution table is:

$x_1 \setminus x_2$	0	3
0	$0.4 \times 0.4 = 0.16$	$0.4 \times 0.6 = 0.24$
3	$0.6 \times 0.4 = 0.24$	$0.6 \times 0.6 = 0.36$

Equivalently, the joint probability distribution table is

$$f(x_1, x_2) = \begin{cases} 0.36, & (x_1, x_2) = (3,3), \\ 0.24, & (x_1, x_2) = (3,0) \text{ or } (0,3), \\ 0.16, & (x_1, x_2) = (0,0), \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $\bar{X}_2$ be the average score over 2 games. Find the probability mass function of $\bar{X}_2$.

Solution:

$$f(x) = \begin{cases} 0.16, & x = 0, \\ 0.48, & x = 1.5, \\ 0.36, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(3) Find the expectation $E[\bar{X}_2]$ and variance $Var(\bar{X}_2)$.

Solution:

$$E[\bar{X}_2] = E[X_1] = 3 \times 0.6 + 0 \times 0.4 = 1.8.$$

$$Var(X_1) = E[X_1^2] - (E[X_1])^2 = 2.16, \quad Var(\bar{X}_2) = \frac{Var(X_1)}{2} = 1.08.$$

(4) Suppose the two teams play 54 games. Approximate the probability that Team A's average score is at least 2.

Solution:

Since the sample size $n = 54$ is large, by the Central Limit Theorem:

$$\bar{X}_{54} \sim N\big(1.8, \tfrac{2.16}{54}\big).$$

Equivalently,

$$Z = \frac{\bar{X}_{54} - 1.8}{\sqrt{2.16/54}} \sim N(0,1).$$

Therefore,

$$P(\bar{X}_{54} \geq 2) = P\left(\frac{\bar{X}_{54} - 1.8}{\sqrt{2.16/54}} \geq \frac{2 - 1.8}{\sqrt{2.16/54}}\right) = P(Z \geq 1) \approx 0.1587.$$