Introduction to Statistics - Quiz #2(60 minutes)

October 14, 2025 (Tuesday)

Section(교반): _____ Cadet Number(교번): _____________ Name(성명): __________________ Score: ____________

- All solutions must include a detailed step-by-step explanation.
- If an answer has more than four decimal places, round to the **fourth decimal place**.
- **Reference table** is provided on the last page of the exam.

1. Jisoo's waiting time for bus X, measured in minutes, is uniformly distributed on the interval $(0, 15)$. [25 points]

(1) Find the probability density function(pdf) of X.

Solution: Since X is uniformly distributed on the interval $(0, 15)$, its probability density function is

$$f(x) = \begin{cases} \frac{1}{15} & 0 < x < 15, \\ 0 & \text{otherwise.} \end{cases}$$

(2) What is the probability that Jisoo's waiting time X is less than 3 minutes?

Solution:

$$P(X < 3) = \int_{-\infty}^{3} f(x)dx = \int_{0}^{3} \frac{1}{15}dx = \frac{3}{15} = \boxed{0.2}.$$

(3) Find the expectation and variance of X.

Solution:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{15} x \cdot \frac{1}{15}dx = \frac{15}{2} = \boxed{7.5}.$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - 7.5)^2 f(x)dx = \int_{0}^{15} (x - 7.5)^2 \cdot \frac{1}{15}dx = \frac{1}{15}\left[\frac{1}{3}(x - 7.5)^3\right]_0^{15} = \frac{2 \times 7.5^3}{15 \times 3} = \boxed{18.75}.$$

In general, if X is uniformly distributed in (a, b), then $E(X) = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$.

2. Suppose the time required to complete a task follows a normal distribution with mean 30 minutes and standard deviation 3 minutes. [25 points]

(1) Find the probability that the task completion time falls between 27 minutes and 36 minutes.

Solution: Let a random variable X denote the task completetion time. Then $X \sim N(30, 3^2)$. Therefore,

$$\begin{aligned} P(27 \le X \le 36) &= P\left(\frac{27-30}{3} \le \frac{X-30}{3} \le \frac{36-30}{3}\right) = P(-1 \le Z \le 2) \\ &= P(Z \le 2) - P(Z \le -1) = 1 - P(Z \le -2) - P(Z \le -1) \\ &\approx 1 - 0.0228 - 0.1587 = \boxed{0.8185}. \end{aligned}$$

(2) What is the cutoff for the top 20% of task completion times?

Solution: We want the 20th upper percentile of $X \sim N(30, 3^2)$. That is, find x such that $P(X \ge x) = 0.20$.

$$P(X \ge x) = P\left(\frac{X-30}{3} \ge \frac{x-30}{3}\right) = P\left(Z \ge \frac{x-30}{3}\right) = 0.20.$$

Since this is the right-tail area of standard normal distribution, $\frac{x-30}{3} = z_{0.20} \approx 0.8416$. Thus,

$$x = 30 + 0.8416 \times 3 = \boxed{32.5248}.$$

3. In a university survey of 800 students, 440 reported that they smoke. Construct a 95% approximate confidence interval for the population proportion of students who smoke. (Assume that conditions for the Central Limit Theorem are satified.) [20 points]

Solution:

We use: $z_{0.025} = 1.96, \quad \hat{p} = \dfrac{440}{800} = 0.55, \quad SE = \sqrt{\dfrac{0.55 \times (1 - 0.55)}{800}} = 0.0176.$

The 95% approximate confidence interval is:

$$\hat{p} \pm z_{0.025} \cdot SE = 0.55 \pm 1.96 \cdot 0.0176 \approx \boxed{(0.5155,\ 0.5845)}$$

4. About 20% of adults worldwide drink *two or more* cups of coffee daily. To test whether the proportion of such daily coffee drinkers in South Korea differs from the global proportion $p_0 = 0.2$, a random sample of 100 adults in South Korea was surveyed. We conduct **a two-sided hypothesis test for a proportion** using a significance level of $\alpha = 0.05$. [30 points]

(1) State the null and alternative hypotheses.

Solution:

$$H_0 : p = 0.2, \qquad H_A : p \neq 0.2$$

(2) Find the null distribution of the test statistic. Assume that conditions for the Central Limit Theorem are satified.

Solution:

Under H_0, the sampling distribution of $\hat{p}$ is: $\hat{p} \dot\sim N\left(0.2, \dfrac{0.2(1-0.2)}{100} = 0.04^2\right)$

or equivalently, $Z = \dfrac{\hat{p} - 0.2}{0.04} \dot\sim N(0, 1)$

(3) In the sample of 100 people, 28 reported drinking two or more cups daily. Compute the observed test statistic.

Solution:

$$\hat{p} = \frac{28}{100} = \boxed{0.28}, \text{ or } z = \frac{0.28 - 0.2}{0.04} = \boxed{2}$$

(4) Using the test statistic in (3), compute the p-value and complete the hypothesis test. State the conclusion in the context of data.

Solution:

$$p\text{-value} = 2P(Z > 2) \approx 2 \times 0.0228 = \boxed{0.0456}$$

Since the p-value is smaller than $\alpha = 0.05$, we reject the null hypothesis. There is evidence that the proportion of daily heavy coffee drinkers in South Korea differs from 20%.

(5) Consider the p-value as a function of the sample proportion $\hat{p}$. What value of $\hat{p}$ maximizes the p-value?

Solution: The two-sided p-value is maximized when $z = \dfrac{\hat{p} - 0.2}{0.04} = 0$, i.e., at $\boxed{\hat{p} = 0.2}$.

Reference Table

$z_{0.025} = 1.9600$	$z_{0.05} = 1.6449$	$z_{0.1} = 1.2816$	$z_{0.2} = 0.8416$
$P(Z \leq -2) = 0.0228,\ Z \sim N(0,1)$		$P(Z \leq -1.7817) = 0.0374,\ Z \sim N(0,1)$	
$P(Z \leq -1) = 0.1587,\ Z \sim N(0,1)$		$P(Z \leq 0.2) = 0.5793,\ Z \sim N(0,1)$	
$P(Z \leq 0.28) = 0.6103,\ Z \sim N(0,1)$		$P(Z \leq 0.55) = 0.7088,\ Z \sim N(0,1)$	