Statistical Methods - Quiz #2(70 minutes)

April 8, 2026 (Wednesday)

Section(교반): __A1__ Cadet Number(교번): __________ Name(성명): ______________ Score: ____________

- All solutions must include a detailed step-by-step explanation.
- Reference sheet is available on the last page.

1. Consider the logistic regression model with a single binary covariate $x_i \in \{0, 1\}$ but without intercept:

$$\log \frac{p_i}{1-p_i} = \beta_1 x_i, \quad y_i \sim \text{Bernoulli}(p_i) \text{ independently}, \quad i = 1, \cdots, n.$$

The data are summarized in the following contingency table:

		Y		
		0	**1**	**Total**
X	0	n_{00}	n_{01}	$n_{00}+n_{01}=n_0$
	1	n_{10}	n_{11}	$n_{10}+n_{11}=n_1$
	Total	$n_{00}+n_{10}$	$n_{01}+n_{11}$	$n_{00}+n_{01}+n_{10}+n_{11}$

(A) Show that

$$p_i = \frac{1}{1+\exp(-\beta_1 x_i)}.$$

(B) For subjects with $x_i = 0$, $p_i = p(0) = \dfrac{1}{2}$; for $x_i = 1$, $p_i = p(1) = \dfrac{e^{\beta_1}}{1+e^{\beta_1}}$. Express the odds ratio(OR)

$$\text{OR} = \frac{p(1)/(1-p(1))}{p(0)/(1-p(0))}$$

in terms of β_1.

(C) Show that

$$E(n_{00}) = E(n_{01}).$$

Hint: You may use $E(n_{00}) = E\left(\sum_{i=1}^{n}(1-y_i)(1-x_i)\right) = \sum_{i=1}^{n} P(y_i = 0 \mid x_i = 0)(1-x_i)$, and $E(n_{01}) = E\left(\sum_{i=1}^{n} y_i(1-x_i)\right) = \sum_{i=1}^{n} P(y_i = 1 \mid x_i = 0)(1-x_i)$.

(D) Show that the closed form MLE(Maximum likelihood estimator) $\hat{\beta}_1$ that solves the score equation

$$S(\beta_1) = \sum_{i=1}^{n}(y_i - p_i)x_i = 0$$

is $\hat{\beta}_1 = \log \dfrac{n_{11}}{n_{10}}$

(E) The Fisher information is given as

$$\mathcal{I}(\beta_1) = \sum_{i=1}^{n} p_i(1-p_i)x_i^2 = \sum_{i=1}^{n} p_i(1-p_i)x_i.$$

Show that the asymptotic variance of $\hat{\beta}_1$ evaluated it at the MLE is

$$\widehat{\text{Var}}(\hat{\beta}_1) = \frac{1}{n_{10}} + \frac{1}{n_{11}}.$$

(F) Consider the hypothesis test:

$$H_0 : \beta_1 = 0, \quad H_A : \beta_1 \neq 0.$$

Show that the Wald test statistic is

$$W = \hat{\beta}_1 \times \mathcal{I}(\hat{\beta}_1) \times \hat{\beta}_1 = \sum_{i=1}^{n} \hat{p}_i(1-\hat{p}_i)\left(\log \frac{\hat{p}_i}{1-\hat{p}_i}\right)^2,$$

where $\hat{p}_i = p_i(\hat{\beta}_1)$. What is the null distribution of W?